PRESCRIPTIVE KINSHIP SYSTEMS, PERMUTATIONS, GROUPS AND GRAPHS

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Abstract

We show how mathematical methods may be applied to the description of all potential prescriptive kinship systems, allowing us to classify them and to understand why some marriage rules are more frequent than others. The modeling will successively use permutations for representing filiation in direct line, group theory to model the whole system, and graphs to help in enumerating possible cases.

1. Prescriptive kinship systems.

A *prescriptive kinship system* (hereafter referred as PKS) is the partition of a society in classes, that is, the mention of a list of names from which one and only one will be given to any member of the society, accompanied by a series of *prescriptive marriage rules* of the kind:
'a man from class *x* shall take as spouse a woman of class *y* and their children will pertain to class *z*'.
Note that no hierarchy is created between classes, which distinguishes such a system from a cast system

More generally, there exist *proscriptive kinship systems*, that is, systems where marriage is restricted by rules of the type ' a man from class *x* shall not take as spouse a woman of class *y*'. Prescriptive systems can be taken as the most restrictive case, where all classes but one are excluded for any individual. [LEH]

The 'class identity' is quite strong and is reflected in the fact that kinship terms used by an individual are the same for all persons of the same class and the same sex. For example, since those rules imply that brothers always pertain to the same class, the word for 'father' will also be used for 'father's brother' (see [LIU], pp. 1-12 or [TSF], pp. 34-49 for more details) .

Such systems have been called Aboriginal class structures, but are more widespread than that.

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They are widely encountered in Australia (e.g. Kariera, Arrente), Melanesia (Vao, Ambrym, Malekula) and South-East Asia (Tamil-Telugu).

1.1. Balanced systems

A PKS will be called *balanced* if the rules allowing or disallowing marriage between relatives are the same for every person of a given sex, that is if, whenever some individual is allowed (resp. disallowed) to marry some specific relative, all same-sex individuals in the society are allowed (resp. disallowed) to marry a person which has the same relation to them, regardless of class.

For example, in the Arrente (Aranda, Arunta) society of Central Australia, every man is allowed to marry the sister of his sister's spouse, and no man is allowed to marry any cousin.

Most PKS are balanced.

1.2. Connected systems

A PKS will be called *connected* if, given any two individuals in the society, the rules allow them to have a common descendant.

Not all PKS are connected. For example, the well-known case of the Murngin society is composed of two *connected components*, that is, two sub-societies such that a person in the first and a person in the second will never have a common descendant, but which function separately as connected systems.
However, for the sake of our classification, it is by no means restrictive to limit ourselves to connected PKS. One may separately study every component of a disconnected society.

So, from now on, we will only consider balanced and connected PKS.

1.3. Types

It will be useful to shift our approach from *classes* to *types*.

A *type* will be made of all the males of one class and all the females of another class which are allowed to intermarry.
Types create a partition, as classes do.

The marriage rules will then be summed up as 'a male and a female are allowed to marry if and only if they pertain to the same type'.
However, the children of a given couple will now pertain to two different types : one for the females, one for the males.

The remainder of this study will consider the society as partitioned in types rather than in classes.

2. Permutations.

Let S be a PKS, with types $t_1 \ldots t_n$.

The masculine filiation determines a permutation on the types : each type is mapped to one and only one other type by the permutation 'having its sons of type …'
Another permutation is generated by the feminine filiation.
Call those permutations *m* and *f* respectively.

We may now translate the balancedness condition and the connectedness conditions into conditions on *m* and *f*.

A PKS is connected if and only if, for given types t_i and t_j, there exists one permutation, product of *m*, *f* and their reciprocals, in any number and order, that maps t_i onto t_j. Actually, let the common postulated descendant be of type t_k ; then the 'path' from t_i to t_k is composed of several occurrences of *m* and *f* and the path from t_j to t_k of another series of occurrences of *m* and *f*. (it is *generated* by *m* and *f*)
Composing the first series with the reciprocal of the second creates a permutation of the types (a product of permutations is a permutation) which maps t_i onto t_j.

$\alpha_1\alpha_2\ldots\alpha_r\ (t_i) = t_k$
$\beta_1\beta_2\ldots\beta_s\ (t_j) = t_k$
$\alpha_1\alpha_2\ldots\alpha_r\beta_s^{-1}\ldots\beta_2^{-1}\beta_1^{-1}\ (t_i) = t_j$
where the α's and β's are either *m* or *f*.

The balancedness of S implies that the only permutation generated from *m* and *f* that maps a type onto itself is the identity permutation.

To convince ourselves of this fact, we can observe that any *relation* stating that some product of *m* and *f* maps t_t onto itself defines permitted marriages ; for example, from *mm* $(t_t) = t_t$ comes that a female of type t_t may marry her sister's grandson, since they will be of the same type (the fact that ages may make this difficult is irrelevant here), and reciprocally the fact that this marriage is possible implies that *mm* $(t_t) = t_t$. Since the PKS is balanced, this has to be true for every female individual. And thus *mm* $(t_k) = t_k$ for every type t_k, that is, *mm* is the identity permutation.

What's more, given any types t_i, t_j, among all permutations generated by *m* and *f*, only one maps t_i onto t_j.. For if there were two such permutations, say *p* an *q*, the product permutation $p.q^{-1}$ would map t_i onto itself, and as we've seen above, it would be the identity permutation. From $p.q^{-1} = I$ it follows that $p = q$. So there may exist only one such permutation.

To sum up:
1. To every balanced, connected PKS may be associated two permutations *m* and *f* , acting on the types, which represent the masculine and feminine filiation.
2. The set of permutations generated by *m* and *f* , that is the set of all products of repeated occurrences of *m* and *f* in any order, contains, for any two given types, one and only one element mapping the first type onto the second. This set will be said *primitive* on the types.

2.1. Incest prohibition.

All PKS have as a rule the prohibition of incest between brother and sister.
To ensure that, it suffices that the sons and daughters of any given type do not pertain to the same type.

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In terms of permutations, it translates as $m\ (t_i) \neq f\ (t_i)$ for every t_i , which will be satisfied whenever *m* and *f* are not the same permutation (saying this is logically equivalent to stating that $m\ (t_i) = f\ (t_i)$ implies $m = f$, which is abovementioned rule #2).

This condition is not necessary to the developments that follow, thence it can be discarded, so that, if there ever is discovered a PKS that does not prohibit incest, the validity of the conclusions will not be altered. But $m = f$ only gives rise to fairly trivial cases anyway.

3. Groups.

It is well known that the set of all permutations generated from a given set of permutations on the same elements, with the composition as an operation, constitutes a *permutation group* in the mathematical sense of the term, that is, it satisfies the following conditions :
- the set is closed (the result of the operation on any elements of the set lies inside the set)
- the operation is associative (this is always true of the composition of permutations)
- the identity permutation (noted I) belongs to the set
- the reciprocal of any permutation belonging to the set belongs to the set

The last two conditions may be replaced by the condition that any equation $p\ .\ x = q$, where *p* and *q* are elements of the set, has one solution *x* belonging to the set.

(The argument that 'from $p.q^{-1} = I$ it follows that $p = q$.', as used before, is not valid when *p* and *q* are elements of a random set, together with a random operation. But it is true within groups, which gives it its legitimacy)

We now have the very important statement :

Every balanced, connected PKS may be characterized by the following data :
1) *some permutation group ;*
2) *the data of two specific elements of the group, that together generate the whole group ; those will represent permutations m and f.*

What's more, the order (or cardinality) of the group is the same than the cardinality of the set of types, since given some type t_t there exists, for every type t_j, exactly one permutation mapping t_i onto t_j, that is, there exists a bijection between types and elements of the permutation group.

For example, to the Arrente society, as described in [LST], *passim* , and [LIU], pp. 201 *sqq* , is associated the group D_8 (the isometry group of the square), and *m* and *f* are represented in this group by a 90° rotation and a mirror, respectively, as will be shown below.

The utility of this is that the classification of possible PKS can be rested on the classification of groups of small order, which is well established (see for example [DEL]).

Note that several groups will never be associated with PKS, since it isn't true that every group can be generated by two well-chosen elements, and we need the group to be generated by *m* and *f*.

4. Graphs.

Given a permutation group, with its generating permutations, acting on a given set, it can be represented by its *Cayley diagram*, that is, a graph whose vertices are the elements of the set, with distinct sets of arrows (we'll call them *quivers*, although this term is nonstandard), one quiver for each generating permutation.

This means we will be able to represent the group associated to every PKS in the form of a graph with as many vertices as there are types in the PKS, and two quivers representing the effects of masculine and feminine filiation respectively. We will speak of *m-arrows* and *f-arrows.*

Any possible marriage will thence be readable directly on the graph : starting from one vertex representing the common ancestor of two related persons, the paths representing their respective filiations will end up at the same vertex, meaning the two persons will belong to the same type. Note that the paths will have to end, one with an *f*-arrow, the other with a *m*-arrow, because they have to represent filiations of persons of different sex.
More generally, the fact that two given relatives of some person do or don't belong to the same type will be easily checked on the graph: follow the paths corresponding to both relationships. If they end at the same vertex, the two persons indeed belong to the same class. If they do not, they do not. And, since we are dealing with balanced PKS, the answer will be the same whichever person (with specified sex) we started from.

This can in turn be translated into a *relation* between products of *f* and *m*, stating a groupal property.
The shuttling between filiation, its graph representation, and corresponding relations is the core of kinship systems study.

The balancedness condition translates into the condition that the graph be vertex-regular, that is, for every vertices v_i, v_j, there exists an automorphism of the graph (permutation of its vertices mapping every arrow onto an arrow of the same quiver) which maps v_i onto v_j. In mundane words, the graph 'looks alike' when starting from every vertex.

The connectedness translates into the condition that the graph be strongly connected, that is, every vertex may be joined to every other vertex by a path of consecutive arrows, taking into account the orientation of the arrows (not necessarily belonging to the same quiver).

This will be of considerable use, since the balancedness and connectedness properties are easily checked on a graph; we will then be able to construct possible PKS in an exhaustive way by constructing their associated graphs.

5. Parameters of a PKS.

We will need an easy way to characterize and classify possible PKS.

The first idea that springs to mind is to classify according to the number of types (which is the same as the number of classes). But this classification is too coarse.

An easy result from group theory states that, in a finite group, given any element *e*, there exists some power of *e* (that is, the product of *e* by itself some finite number of times) equal to

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the identity element : there exists k such that e^k = I. *e* will then be said *of order k*, if *k* is the smallest such integer.

This means that, in the group associated to a PKS, *m* and *f* will have some well-defined orders, not necessarily the same for *m* and *f*.

It appeared that the classification of PKS according to their number of types, the order of *m* and the order of *f* was quite effective and not too difficult to establish.
We will thus characterize every PKS by a triplet of integers : (T, M, F), respectively the number of types, the order of M and the order of F.

Now call 'I' some vertex of the Cayley graph (the fact that the graph is vertex-regular means that we don't have to consider which). Then every vertex can be named by the (sole, remember) permutation which maps I onto it.

Now, the circuit of *m*-arrows containing vertex I will have as members I, *m*, m^2, … and there will be M of them, because *m* is of order M. (result 5.1)
Since we could have begun with any vertex, every vertex belongs to one circuit of *m*-arrows (we'll speak of *m-circuits*). It also belongs to only one, since there aren't two *m*-arrows starting from a given vertex.

So, the set of vertices is partitioned in *m*-circuits, and they will all be of the same length (a consequence of balancedness). Thus M must be a divisor of T.[3] (result 5.2)

Of course, the same argument holds for F.
This means we will have to consider only a relatively small number of values for M end F for a given value of T, which is a good thing when aiming at a complete classification. It also means that the more divisors T has, the most intricate the classification of PKS with T types will be.

Note that it is quite possible that there exist no possible Cayley graph with given parameters T,M,F, even with M and F divisors of T. Also, it is possible that for a given triple of parameters T,M,F there exist more than one possible Cayley graph, but the number of those will be small, since the connectedness and balancedness conditions are quite restrictive.

6. Some results that may help.

6.1. We have seen that M and F must divide T. Thus, if T is prime, they have to be either 1 or T. It is impossible that both M and S be equal to 1, for thence *m* and *f* would both be the unit permutation. In this case, not only would we have *m* = *f* (which isn't a redhibitory defect, but creates incest), but furthermore the graph wouldn't be connected, thus the PKS wouldn't either. Thus, we have :
If T is prime, either M or S (or both) must be equal to T. If one isn't, it is equal to 1.

6.2. More generally,
If M is equal to 1, then F is equal to T, and conversely.

[3] Those who are versed in group theory will recognize the classical Lagrange theorem : the *m*-circuit containing I constitutes a subset of the permutation group, the other *m*-circuits are its cosets, and the order of the subgroup (and cosets) divides the order of the group.

The argument is basically the same as in 6.1. : if M were equal to 1 and S to some number smaller than T, then the graph would not be connected, and neither would be the PKS.

6.3. If either M or F is equal to T, then the associated group is cyclic.
For thence all elements would be powers of *m* (result 5.1) and only cyclic groups have this property.
In particular, F is some power of M (and conversely). We can thence further refine the classification of PKS with parameters (T,T,T) (that is, both *m* and *f* are of order T), according to the number *x* such that $m^x = f$, which is quite useful, since such PKS are more numerous then for other given triplets of parameters .

6.4. If T=M=F, x (as defined above) *and T must be relatively prime.*
If *x* and T had a common divisor, say *z*, one would have both
$x = a.\,z$ and $T = b.z$
From there we can derive :
$f^b = m^{bx} = m^{baz} = m^{Tz} = (m^T)^z = I^z$
(the last step is because *m*'s order divides T)
So, some power of *f*, smaller than T, will be the identity, which contradicts the statement that T=F.

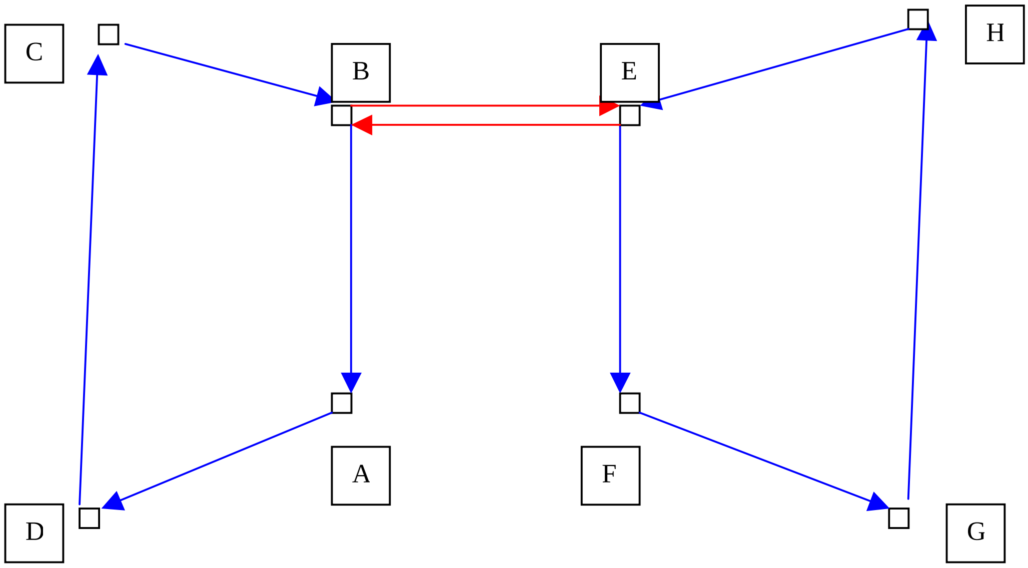
6.5 If two or more m-circuits include vertices from a same f-circuit, they contain the same number of those.
Otherwise, the balancedness property would not hold.
This excludes several combinations of parameters, for example T = 2*a*, F = M = *a*, where *a* is odd.

7. An example.

We will now show how working on graphs helps in enumerating possible PKS.
Take the example T = 8, M = 4 , F = 2.
The Cayley graph will thence be made of 2 *m*-circuits of length 4, and 4 *f*-circuits of length 2.

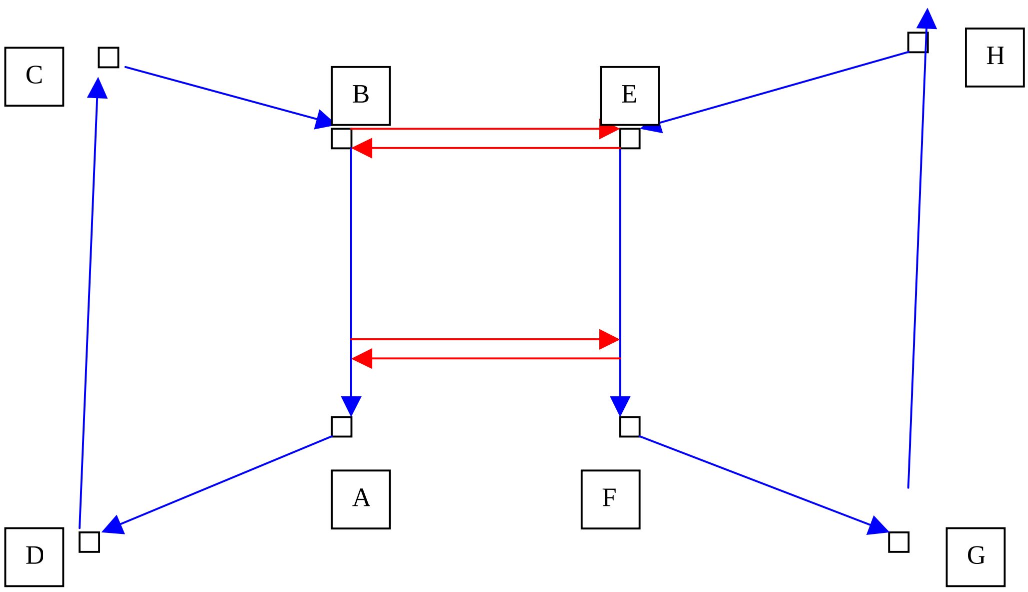
We may draw the *m*-circuits and one 2-circuit to begin with[4]:

[4] We'll use the convention of blue for *m*-arrows and red for *f*-arrows, which seems more egalitarian than the classical convention of plain and dashed arrows.



We mapped a vertex belonging to one *m*-circuit onto a vertex belonging to the other, for else the balancedness rule would force that no *f*-arrow go from one *m*-circuit to the other, and the graph wouldn't be connected. Now, vertex A can be mapped to any of the 3 'free' vertices on the right part of the graph. This will give rise to three subcases :

a)



Hence *f* will have to map H onto C, and conversely, because the fact that *f.m* and *m.f* both map H onto B implies that the images of any vertex by *f.m* and *m.f* will be the same. This follows from the balancedness condition. We will simply write the relation *f.m* = *m.f*.
And, by the same token, *f* maps G onto D and conversely. This gives us complete description of *m*'s and *f*'s action, and of the society's properties (in terms of allowed marriages).

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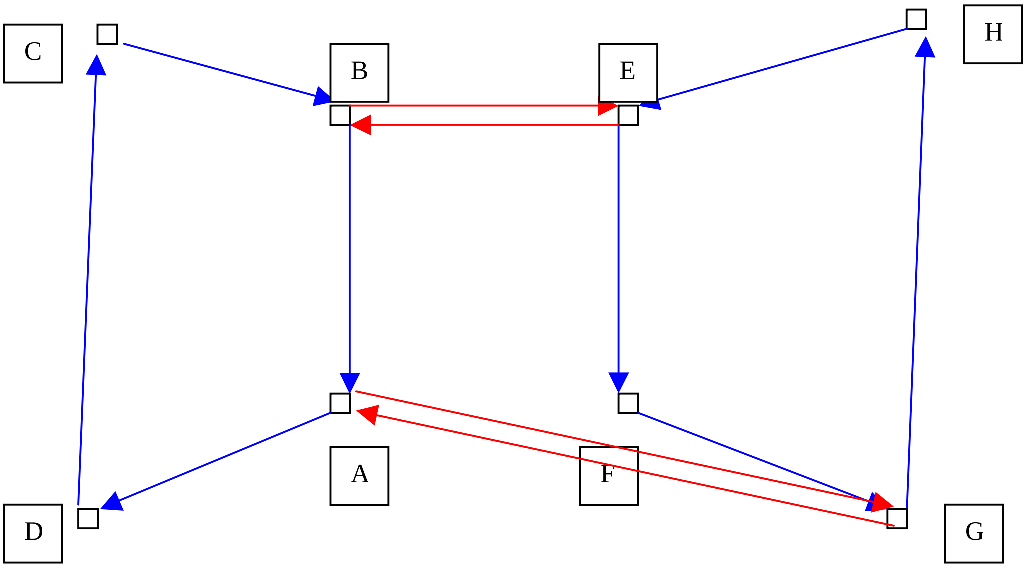
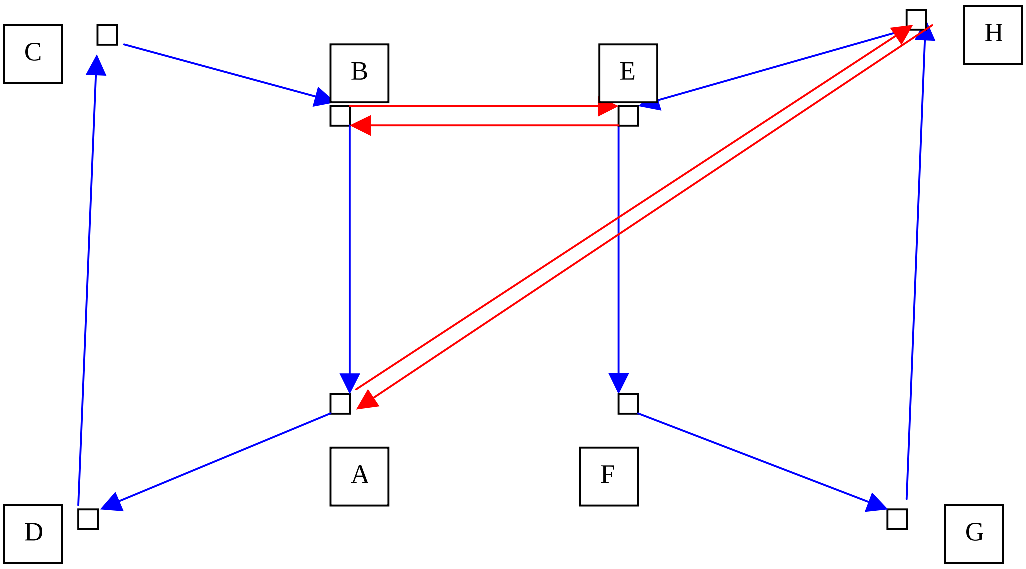
b)



Since *fmm* = *mf* holds for vertex B, it has to hold for any vertex, in particular for vertex H. Hence this graph may only be completed by mapping H onto C, in such a way that both corresponding paths run from H to A. Hence F and D have to be mapped onto each other. But this is not possible, because *fmm* = *mf* will not be true for vertex F (the *fmm* path ends at B, while the *mf* path ends at A).

In short, mapping A onto G leads to a contradiction, and this case may be discarded as incompatible with the balancedness condition.

c)



Now, for vertex B, $fm^3 = mf$. This must be true for all other vertices, which means that *f* has to map G onto D, to ensure that this same relation will be true for vertex G (both corresponding paths ending at A). Now C and F will be mapped onto each other.

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Once again, we have obtained complete description of *m*'s and *f*'s action, and of the society's properties (in terms of allowed marriages). It happens that this is the graph of the Arrente system.

We've now proven that there may only exist two different (ie with different allowed marriages) societies with parameters T = 8, M = 4, F = 2.

We may also identify the associated groups:

In case a), the group must be Abelian (since $fm = mf$, and the commutativity of generators implies the commutativity of any two elements in the group). Also, it has exactly three elements of order 2 (f, fm^2 and m^2), as can be seen by following corresponding paths on the graph. This unequivocally characterizes the group as C4×C2, the subgroup of the isometry subgroup of a quadratic prism generated by a 90° rotation and the central symmetry.

In case c) the group is non-Abelian (since $fm \neq mf$), and has more than one element of order 2 (in fact, it has five : f, m^2, fm, fm^2, fm^3). This unequivocally characterizes the group as D8, the isometry group of the square. See for example [WEI] or [DEL].

We can also add some statements about relatives. For example, we are able to read, directly on the graph for case c), that $f.m.f = m$. This translates into the fact that the son of any person is of the same type than the daughter of his/her daughter's son.
Some relations have meaning only for males or only for women. As an example, once again consider graph c) and remark that $g^{-1}.f = f.g$, which means that, for every male, his sister is of the same type that his daughter's son.

8. Some general results in terms of allowed marriages.

8.1. Marriage between direct cousins.

There are four possible marriages between direct cousins:

a) a male marries his patrilateral parallel cousin, the daughter of his father's brother.
b) a male marries his matrilateral parallel cousin, the daughter of his mother's sister.
c) a male marries his patrilateral cross cousin, the daughter of his father's sister.
d) a male marries his matrilateral cross cousin, the daughter of his mother's brother.

Cases a) and b) never happen in PKS, while case c) is uncommon and case d) is quite common.
This can be explained by the use of relations: starting from the nearest common ancestor, which happens to be the common grandparents, those marriages may be represented by the following relations :
a) $mm = mf$
b) $fm = ff$
c) $mm = ff$
d) $mf = fm$

Relations a) and b), in a group, reduce to $f = m$, which makes incest between brother and sister possible. So, in any PKS where this form of incest is disallowed (and this is the case of any known PKS), those two marriages between cousins will never happen.
Relation c) will occasionally happen between two given elements of a group, making this marriage possible.
But relation d) will be true whenever the group is Abelian, which is the case of most groups of small order. Therefore, for a random PKS, the probability that marriage d) happen is quite high. In particular, it will always be possible in PKS whose number of types is either 4 or any prime integer, since groups of order 4 or any prime are Abelian.

8.2. The case where T = M.

When T = M, the group must be cyclic, as seen in 6.3.
Hence f is some power of m. As in 6.3, let $f = m^x$.
Since a cyclic group is Abelian, the marriage between a male and his matrilateral cross cousin will always be possible.

Now consider the two following marriages :
a) between a male and his patrilateral cross cousin, which as seen in 8.1 translates into $mm = ff$
b) between a male and the sister of his sister's spouse, which translates into $m = f^{\cdot}m^{-1}.f$ (starting both paths from the male's parents).

$mm = ff$ becomes in this case $m^{2x} = mm$, or $m^{2x-2} = I$. This is the case only when $x = (M/2)+1$, or $x = (T/2)+1$; since by hypothesis T = M.

On the other hand, $m = f^{\cdot}m^{-1}.f$ becomes $f^{-1}.m = m^{-1}.f$, or $m^{-x}.m = m^{-1}.m^{x}$, or $m^{x-1} = m^{1-x}$.
Since by the definition of x, it may not be equal to 1, it follows that
$(1-x) = (x-1) = M/2$
which means that $x = (M/2)+1$.

We thence have the following result :
When T = M, marriages a) and b) are simultaneously possible or simultaneously impossible.
Also, since relations given in *a* and *b* are unchanged when exchanging the roles of f and m, this is also true when T = F.

Conclusion.

Simultaneous consideration of a PKS's rules, relations in the associated group, and its representation by a Cayley graph, and the constant shifting between these three views, are the easiest way to :
- enumerate possible PKS ;
- identify a PKS with one of the listed cases ;
- determine classes of PKS which satisfy some specific marriage rule ;
- represent a specified PKS in such a way that one may unambiguously refer to it, namely its associated Cayley graph.

which gives us many tools to understand the mechanics of PKS.

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