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COMMENT ON GIL'S *"WHAT ARE THE BEST HIERARCHICAL ORGANIZATIONS FOR THE SUCCESS OF A COMMON ENDEAVOUR?"*

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COMMENT ON GIL'S "WHAT ARE THE BEST HIERARCHICAL ORGANIZATIONS FOR THE SUCCESS OF A COMMON ENDEAVOUR?"

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Abstract: In his study "What are the best hierarchical organizations for the success of a common endeavour" [*Mathematical Anthropology and Cultural Theory* 9, 1 (2016)] Gil uses coupled oscillator networks as a model for socially interacting individuals. In this commentary, I argue that such networks could also be related with lattice gauge theory. The most stable lattice gauge society would be an open society whose gauge invariance indicates the existence of universal values, such as human rights.

In his study "What are the best hierarchical organizations for the success of a common endeavour?", Gil (2016) uses coupled oscillator networks as a model for socially interacting individuals. His important findings are 1) synchronization is a nice mathematical metaphor for common endeavor, 2) linear stability analysis of "social" oscillator networks favors completely connected ("democratic") networks over linear command and obedience ("military") chains, 3) most stable hierarchically organized networks could decompose into "mainstream" and "hipster" sub-cultures.

In my commentary, I would like to point out another mathematical analogy, namely *lattice gauge theory* (Wilson 1974) for additionally strengthening Gil's argument. Looking at the Kuramoto equation (1) in Gil's article, one may regard the current state θ_i of the i-th oscillator as an element of a linear space H that is attached to the graph's vertex i. In the present case of phase angles this state space can be identified with the field of complex numbers $\mathsf{C} \cong \mathsf{R}^2$ such that $\boldsymbol{v}_i = \mathrm{e}^{\mathrm{i}\theta_i}$ is the state vector of vertex i. Without any coupling, $\boldsymbol{A} = 0$, the local oscillators pursue their intrinsic dynamics $\partial_t\theta_i = \omega_i$, i.e. $\theta_i(t) = \theta_i(0) + \omega_i t$. In the new picture, that means $\boldsymbol{v}_i(t) = \mathrm{e}^{\mathrm{i}(\theta_i(0)+\omega_i t)} = \mathrm{e}^{\mathrm{i}\omega_i t}\boldsymbol{v}_i(0)$, which is the *group action* of the unitary group U(1) on the local Hilbert space $\mathsf{H}_i = \mathsf{C}$. Finally, the coupling term requires the comparison of local phases θ_i and θ_j at neighboring graph vertices i, j through $\sin(\theta_i - \theta_j)$. This means, there must be a *connection*, expressed by a *parallel transporter* $\phi[p] : \mathsf{H}_i \to \mathsf{H}_j$ along the edge $p = (i, j)$. Yet these are the basic ingredients for a lattice gauge theory, as illustrated in Fig. 1.

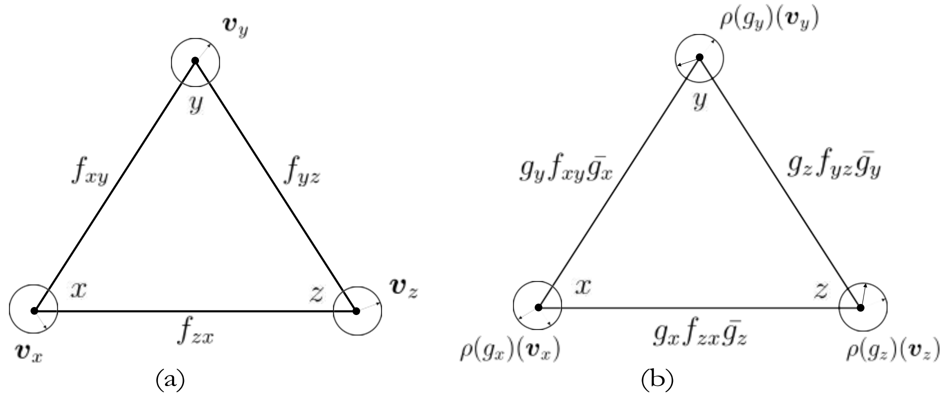

Figure 1: Illustration of lattice gauge theory. (a) Assignment of state vectors $\boldsymbol{v}_x$, $\boldsymbol{v}_y$ and $\boldsymbol{v}_z$ to vertices x, y, z and of group elements f_{xy}, f_{yz}, and f_{zx} to edges of a graph Γ. (b) Assignment after local gauge transformation with group representations acting on state vectors, $\boldsymbol{v}'_x = \rho(g_x)(\boldsymbol{v}_x)$, $\boldsymbol{v}'_y = \rho(g_y)(\boldsymbol{v}_y)$, and $\boldsymbol{v}'_z = \rho(g_z)(\boldsymbol{v}_z)$, and transformed fields $f_{xy}' = g_y f_{xy} \bar{g}_x$, $f_{yz}' = g_z f_{yz} \bar{g}_y$, and $f_{zx}' = g_x f_{zx} \bar{g}_z$. The group inverses are indicated by the bar.

Lattice gauge theory (Wilson 1974) is a powerful discretization technique for numerically solving hard problems of quantum field theory, e.g. the confinement of quarks in quantum chromodynamics. However, abstracting most of the physical peculiarities, we can condense its essence in the following way (cf. Mack (1995)): Given a graph $\Gamma = (V, E)$ with vertices V and edges $E \subset V \times V$, and a group G, we consider a representation ρ of G on a linear space H, such that for a group element $g \in G$, $\rho(g)$ is a matrix acting on H. The lattice gauge field is then given by an assignment of state vectors $\boldsymbol{v}_x \in \mathsf{H}$, attached to a vertex $x \in V$, and of group elements $\boldsymbol{f}_{xy} \in G$ attached to the edges $(x, y) \in E$. Two states $\boldsymbol{v}_x$ and $\boldsymbol{v}_y$ at different sites x and y can then be compared when they are connected through an edge $p = (x, y) \in E$ by applying the parallel transporter $\phi[p]\boldsymbol{v}_x = \rho(f_{xy})\boldsymbol{v}_x$, such that the difference $(\phi[p]\boldsymbol{v}_x) - \boldsymbol{v}_y$ is well-defined. As a special case, we call two local states $\boldsymbol{v}_x$ and $\boldsymbol{v}_y$ *transport equivalent* if $\rho(f_{xy})\boldsymbol{v}_x = \boldsymbol{v}_y$. Because G is a group with neutral element e and inverses $\bar{g}$ for any g ∈ G, parallel transporters can be defined for any path across vertices in the graph. The homomorphism $\phi[p] \circ \phi[q] = \phi[pq]$, with pq the concatenation of paths p, q defines a *connection* in lattice gauge theory. Note that the parallel transporter assigned to a fundamental loop (x, x) must be $\rho(e) = \mathbf{1}$, the unit matrix. However, this is not necessarily the case for any closed path p through the graph. If $\phi[p] \neq \mathbf{1}$ for some closed paths p, the connection is called *frustrated*. In physics, frustration is manifested through the presence of forces. Figure 1(a) depicts a possible assignment for a lattice gauge theory in a triangular graph $V = \{x, y, z\}$. The group G is identified with the unitary rotation group U(1) in the complex number plane, which constitute the local state spaces H.

In lattice gauge theory, a given assignment does not uniquely determine the state of the field. Instead, vertices and edges can be gauged according to the group action in the following way: To each vertex x one attaches a local group element g_x such that the state $\boldsymbol{v}_x$ is transformed into $\boldsymbol{v}'_x = \rho(g_x)(\boldsymbol{v}_x)$. Moreover, the field at the edge (x, y) is transformed according to $f'_{xy} = g_y \boldsymbol{f}_{xy} \bar{g}_x$ This reassignment is called *local gauge transformation* and leaves transport equivalence invariant:

$$\rho(\boldsymbol{f}')\boldsymbol{v}'_x = \rho(g_y \boldsymbol{f}_{xy} \bar{g}_x)\rho(g_x)(\boldsymbol{v}_x) =$$
$$\rho(g_y \boldsymbol{f}_{xy} \bar{g}_x g_x)(\boldsymbol{v}_x) = \rho(g_y)\rho(\boldsymbol{f}_{xy})(\boldsymbol{v}_x) = \rho(g_y)\boldsymbol{v}_y = \boldsymbol{v}'_y$$

Figure 1(b) illustrates the impact of a local gauge transformation where states are locally rotated according to some U(1) actions. An important consequence of gauge transformations is that, even in a highly frustrated connection, for some arbitrary vertices x, y, the associated parallel transporter along the path p connecting x, y can be locally gauged to the identity $\phi[p] = \mathbf{1}$, making states $\boldsymbol{v}_x$ and $\boldsymbol{v}_y$ transport equivalent in this particular setting.

In order to apply this framework to social networks we have to identify the states of the vertices and of the edges. Assuming that each vertex of a graph is occupied by an individuum, we may regard the local states $\boldsymbol{v}_x$ as *belief states* in the sense of dynamic semantics (Gärdenfors 1988, beim Graben 2014), i.e. the system of propositions hold to be true by the person x. This state can be straightforwardly represented as a vector according to Hilbert space semantics (Busemeyer and Bruza 2012). The connection comprised by all possible parallel transporters in the network then corresponds to communication means (Mack 1995). Finally, local gauge transformations guarantee that any two individuals that are connected through a communication channel can compare their particular belief states and are thus able to find compromises on their beliefs. Moreover, the property of local gauge invariance indicates the existence of universal values, such as human rights, that are agreeable by all persons in the network.

I expect that Gil's findings on his "democratic networks" could get further support from lattice gauge theory. The most stable *lattice gauge society* would be a network where every individuum can reasonably communicate with every other one - which is the "open society". Moreover, individuals should be susceptible to local gauge transformations, i.e. they have the freedom to "gauge" their particular belief states by means of perspective-taking for comparison with those of their communication partners. Therefore, people should be open-minded. Fanaticism, by contrast, breaks the local gauge invariance when "frozen thoughts" and dogmatism (Arendt 1971) prevent successful and respectful communication between enlightened persons (Stangneth 2016).

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