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Abstract: The study of an anthropological problem leads to the consideration of a class of permutation groups, endowed with a pair of generators. We study some of its subclasses, and test on them properties, stated as equations involving words, which are relevant for the original problem

1. Prescriptive Kinship Complexes

In many parts of the World, traditional societies use *prescriptive kinship complexes*, that is, they partition themselves into classes (moieties, sections etc.), the affiliation of each individual being determined by one's gender and the class of one's parents, and define rules according to which members of given class and given sex may only have as spouse members of some other class.
An important family of such complexes, occasionally called *Aboriginal complexes*, extends over vast parts of Australia and Melanesia, but does not encompass all systems in that zone[1]. Those complexes satisfy two conditions:
- *connectedness:* any two individuals may, in theory, have a common descendant ;
- *balancedness:* the rules are the same for every individual of given sex, i.e. if some male (resp female) of any class is allowed to marry some specific relative (cousin, daughter of cousin etc.), every male (resp. female), irrespective of his (her) class, is allowed to do so.

Notice that, if the rules allow some marriage between close relatives, this marriage will often be seen to happen. Checking for such possible marriages is therefore significant to the anthropologist. Rules for Aboriginal complexes, and their mathematical translation, have been written down by [KST].

One important consequence of balancedness is that, if some same-gender descendant of a person P has same class as P, that would be true of any person of P's gender.

[1] For example, the Murngin system [JDM] isn't connected.

2. A short history of structures for kinship complexes

The fact that algebraic structures, especially groups and monoids[2], are useful for representing kinship, has been ascertained by André Weil [WEI]. For Aboriginal structures, only groups are needed. Essential elements for their description are three permutations:
- permutation *s* maps class C onto the class where a male element of C takes his spouse ;
- permutation *x* maps C onto the class of their male children ;
- permutation *y* maps C onto the class of their female children.

The group or monoid associated to a complex has as its elements all possible products of those permutations.

From this observation, research has followed three courses:

The first is to try to describe as many existing kinship complexes as possible, using general properties of groups and monoids and descriptions in the language of structural algebra. This is the way followed in detail by [LIU].

The second, focusing on Aboriginal-type complexes, is to use the field anthropologists' description to discover associated groups and obtain properties of those complexes. This is the approach of the classical texts [WHI] and [COU].

The third consists of a small change of point of view, which allows a reduction of needed basic permutations from three to two. This, and the fact that the graph describing the base permutations is a well-studied mathematical object, the *Cayley diagram* of the associated group [GDM, section 3], makes describing complexes easier, without changing the algebraic objects to consider. Furthermore, it wipes out the gender bias that using the male-oriented description introduced in classical descriptions [LST, WHI]. Hinted at in [KST], who also used matrices, this new approach was described in [DEM1 and 3, GDM], which put the emphasis on associated graphs.

Apart from the fact that it is gender-symmetrical, this latter approach has the following advantages:

1. It allows for straightforward testing, on the group's Cayley diagram, of permitted marriages, e.g. cross-cousin marriage and sibling exchange [TSF pp. 189 *sqq.*];
2. The existence of kinship morphisms, as described in [COU] and [TSF], can also be checked on the diagram ;
3. It dispenses with the redundancy of the 3-permutation description ;
4. The description of the associated group is more group-theoretical-like, allowing for easy use of known structural theorems and taxonomical results about groups to produce general results about kinship complexes. The description of possible complexes is now derived from that of possible groups, rather than the opposite.

[2] A monoid is a set endowed with an associative operation * and containing an identity element *e* ; a group is a monoid where every element *x* has a reciprocal element x^{-1} such that $x * x^{-1} = x^{-1} * x = e$.

The aim of this paper is to present some taxonomical results of this third approach.

3. Kinship systems

Start from the classical description of a kinship system (as for example in [LST] and [WHI]) in terms of classes. We replace classes by *types*, a type consisting of the males of one class and the females (from another class) allowed to marry them.

We can now represent an Aboriginal kinship complex by a triplet (T, m, f), where:
- T is a set whose elements are called *types*,
- m (respectively f) is the permutation mapping one person's type onto the type of one's sons (resp. daughters), with $m \neq f$. Note that m and f act on types, not on classes.

We call G the permutation group m and f generate on T (all possible finite products of m and f). On how to construct it, see [CAR], pp. 30-36.

PROPOSITION 3.1: G's action on T is strictly transitive. [DEM 1 and 2, GDM]

PROOF

> Connectedness translates into the fact that, for every elements a, b in T there exists some x in T and α, β in G such that $\alpha(a) = x$ and $\beta(b) = x$.. Hence $\alpha_o\beta^{-1}$ maps a onto b.[3]
> The last remark in section 1 implies that: if α maps a onto itself, it maps every type onto itself, hence it is the identity permutation.
> If there were elements a, b in T and π, ρ in G such that $\pi(a) = b$ and $\rho(a) = b$¸ then $\pi_o\rho^{-1}$ would map a onto itself, hence would be the identity, and $\pi = \rho$. Therefore there exists only one such permutation.⊠

We call (T, m, f, G) a *kinship system*.

A kinship system might as well be seen as a triplet (G, m, f), where G is a primitive permutation group, and $\{m, f\}$ are a (not necessarily minimal) generating subset. However, this approach is less in touch with the original problem of describing the relations between types (or classes) ; we will therefore prefer the former description.

Connectedness and balancedness are easily seen on the Cayley diagram for G, the graph C(G) whose vertices are the elements on which it acts (here, types) and whose edges represent its generating elements (here, m and f): connectedness of the complex becomes plain connectedness for graphs, while balancedness means that C(G) is homogeneous on its vertices.

[3] We use *direct notation* (first element first) for products of permutations.

f and *m* may be seen as the quotient of relations "has as daughter" and "has as son" by the partition of the society into types.

4. Representing a Kinship System

Remark that G being regular means that a bijection may be defined between G and T, by choosing one type *t* and mapping every element *a* of T on the element α of G such that $\alpha(t) = a$. Of course, *t* is mapped on the identity (I).[4] This allows us to call I one vertex of G's Cayley diagram, and name each other vertex by the one element of G that maps I onto it. As C(G) is vertex-homogeneous, the result of this indexation will be the same whichever vertex is chosen as I.

To each marriage (with matrilinear cousin, sibling exchange, with patrilinear once-removed cousin etc.) is associated an equation involving words in G, as shown by André Weil [WEI]. This equation will hold if, and only if, the associated marriage is allowed. As the main point of interest for the anthropologist is the determination of those allowed marriages, we will study whether several such equations hold within this or that system.

The most common equations to be considered are:
(1) $f.m = m.f$: a man is allowed to marry the daughter of his mother's brother ;
(2) $m.m = f.f$: a man is allowed to marry the daughter of his father's sister ;
(3) $f^{-1}m = m^{-1}f$: a man is allowed to marry the sister of his sister's spouse.

Notice that the other two marriages between cousins (son and daughter of two brothers, or of two sisters) give rise to equations $m.m = m.f$ and $m.f = f.f$, which are impossible, since in a group, they reduce to $m = f$. Of course, there are many other such equations and, apart from a few specific cases as the one just mentioned, none can be enforced or excluded *a priori*

Such properties are easily checked on C(G): owing to G being strictly transitive, two words represent the same permutation if and only if both permutations' paths lead from I to the same vertex.

More details, from an anthropological point of view, may be read in [LST, LIU, TSF, GDE, GOT].

5. Parameters for kinship systems

In order to characterize kinship systems, we will use several parameters:
- G's order, noted N;
- the length of cycles of *m*, noted μ (balancedness implies that all cycles have the same length) ;
- the length of cycles of *f*, noted φ (same remark).

[4] For details of how this operation is carried, see [GDM], section 3.

It is obvious that μ and φ are the orders of *m* and *f* in G, and that they must divide N. This is actually a special case of Lagrange's theorem: the number of elements in a subgroup of G must divide that of G (see [CHA], pp. 44-45). It will not be restrictive, in the determination of possible kinship systems and their properties, to consider, from now on, that $\mu \geq \varphi$, especially as equations (1)-(3) are left unchanged after an exchange of *m* and *f*. Also, notice that in the original (anthropological) problem, $\mu > \varphi$ is more frequent than the reverse.

We may therefore assign to each kinship system a label, with the form (N, μ, φ), but one should note that different systems may have the same parameters, e.g. there are two systems with parameters (8,4,2): one (the Karajeri system) [5] has group $C_4 \times C_2$ (direct product of cyclic groups), with *m* an element of order 4 and *f* an element of order 2 other than m^2 ; the other (the Arunta-Warlpiri system) has D_8 (diedric group), with (in its view as the square group) *m* a rotation of ¼ of a turn, and *f* a symmetry with respect to any axis. However, the number of systems sharing a given triplet of parameters will be small (we didn't encounter any case where more than three systems did).

6. Abelian groups

Equation (1) in section 4 means that the generators of G commute. In such a case, any two elements commute: the group is *Abelia*n.
We have at our disposal a classification of Abelian groups: each such group is the direct product of cyclic groups (see [CAR], pp. 98-103).
Since G must be generated by *f* and *m*, we need only consider groups which are either cyclic or a product of two cyclic groups, e.g. the parallelepiped group $C_2 \times C_2 \times C_2$ is excluded.

6.1. Cyclic groups

When μ = N, G includes an element of order N, and is therefore cyclic. But the converse isn't necessarily true. When $\varphi = 1$, μ = N, else G wouldn't be transitive on T (its Cayley diagram wouldn't be connected). But the converse isn't necessarily true. It is also possible that $\mu = \varphi$ = N.

We will therefore consider two classes of kinship systems with cyclic G:
- those with μ = N, *m* having therefore only one cycle on T, which we call *holocyclic* [6];

[5] When a system we describe is encountered in the real world, we will name it by the society(ies) where it is applied. Many such systems are described in [TSF], but please note that this book uses the “classical” descriptions in terms of classes.

[6] Names used in the taxonomy of systems are our coinage.

-those with μ a proper divisor of N (since $\mu \geq \varphi$, it may not be = 1, else G wouldn't be transitive), which we call *staurocyclic*.

6.1.1. Holocyclic systems

Let G be a group with parameters (N, N, φ). Since G is cyclic, and m generates it, f is some power of m. Call this power x. If we fix x, G's Cayley diagram may be unequivocally drawn: m has only one cycle, and one arrow of f spans x arrows of m. The system is therefore determined by the data of N, φ and x:

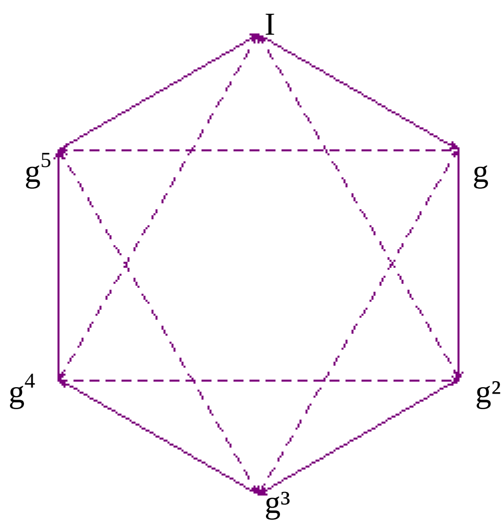

FIGURE 1: a holocyclic system (N = μ = 6, φ = 3, x = 4).
Solid arrows represent the action of m, dashed arrows that of f.

Therefore,

PROPOSITION 6.1

When μ = N, there exists, up to isomorphism, only one kinship system with given N, φ and x.
(its existence may be acknowledged by constructing its Cayley diagram)

THEOREM 6.2

For a holocyclic system, φ = N iff x and N are relatively prime.

PROOF

Suppose that x and N have some common factor $z > 1$.
Hence we may write: $x = a \cdot z$ and $N = b \cdot z$ for some integers a and b.

Note that since $x \leq \mathrm{N}$, $a < \mathrm{N}$.
We have: $f^b = m^{bx} = m^{baz} = m^{\mathrm{Na}} = (m^{\mathrm{N}})^{a.}$
And, since $\mu = \mathrm{N}$, $(m^{\mathrm{N}})^a = \mathrm{I}^a = \mathrm{I}$.
Hence, there exists some power of f, with exponent < N, equal to I, which contradicts φ = N.
Now, say that $\varphi < \mathrm{N}$. Then $f^y = \mathrm{I}$ for some $y < \mathrm{N}$, and $(m^{.x})^{.y} = m^{x.y} = \mathrm{I}$. Since N is the smallest power of m which is equal to I, $y.x$ must be a multiple of N. And since $y < \mathrm{N}$, x must have some common factor with N. ⊠

For a holocyclic system, of course, equation (1) from section 1 always holds. Furthermore, we have:

THEOREM 6.3

For a holocyclic system, equations (2) and (3) hold simultaneously, and this happens iff μ is even and $x = \mu/2 + 1$.

PROOF

Equation (2) may be rewritten as $mm = m^{2x}$.
Since $\mu = \mathrm{N}$, this is true whenever $x = \mu/2 + 1$.
Also, if $mm = m^{2x}$, then $m^{2x-2} = \mathrm{I}$.
Since $2x\text{-}2 < 2\mu$, there follows $2x\text{-}2 = \mu$, and therefore $x = \mu/2 + 1$.
Equation (3) may be rewritten as $m^{-x}.m = m^{-1}.m^x$, or $m^{x-1} = m^{1-x}$.
This is true whenever $x = \mu/2 + 1$. And, since x may not be equal to 1, $m^{x-1} = m^{1-x}$ implies $(1\text{-}x) = (x\text{-}1) = \mu/2$, and therefore $x = \mu/2 + 1$. ⊠

Those equations do hold in the system shown as figure 1.

6.1.2. Staurocyclic systems

These systems have a cyclic group G of order N, but neither m nor f is of order N in G. For this to happen (taking into account that G = <f,m>), G must be a direct product of two cyclic groups of orders μ and φ, which will happen iff μ and φ are relatively prime and $\mu . \varphi = \mathrm{N}$.

Taking $\mu . \varphi < \mathrm{N}$ will not make m and f generators of G, since the order of such a group is the product of its generators' orders. For example, if m is some nonidentity in C_5 and f is of order 3 in C_6, m and f will not generate C_{30}. But they will if f is of order 6. Of course, when μ and φ are prime, picking any nonidentities in C_μ and C_φ will do.

The structure of staurocyclic systems is uniform: m has φ cycles of length μ, f has μ cycles of length φ, and every cycle of m has one element in common with every cycle of f (the fact that, when a cycle of *m* intersects all cycles of *f*, all intersections must have the same cardinality is a straightforward consequence of the balancedness condition). There is one such system for each pair of relatively prime integers.

In such a system, equations (2) and (3) will never hold (the special case where $m = f = 2$ doesn't produce a cyclic system, see figure 2b hereunder).

6.2. Non-cyclic systems

6.2.1. Stauric systems

These are the same as staurocyclic systems, but for the fact that φ and μ aren't relatively prime (they may be equal) and, as a consequence, G isn't cyclic. But all other properties mentioned in section 4.2 remain: existence and unicity for given parameters μ, φ and N, provided that N = μ.φ; the fact that neither (2) nor (3) hold, and the general structure. Actually, stauric and staurocyclic systems are defined by the same relations: $m^a = f^b = mfm^{-1}f^{-1}$; their Cayley diagrams are similar; the difference being that in the case of staurocyclic systems {*m*,*f*} isn't a minimal generating set.

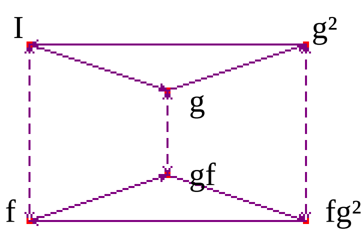

FIGURE 2a
The (6,3,2) staurocyclic system

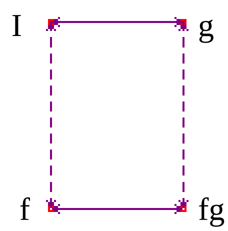

FIGURE 2b
The (4,2,2) stauric system or Kariera system

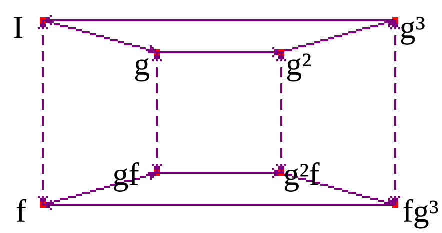

FIGURE 2c
The (8,4,2) stauric system or Karajeri system

The simplest such group is $C_2 \times C_2$, aka V_4, and the corresponding system is known as the *Kariera system* (figure 2b). Another well-documented example is $C_4 \times C_2$, the *Karajeri system* (figure 2c).

The Kariera system is a special case, where (2) and (3) do hold. This is due to the fact that *m* and *f* both are of order 2.

6.2.2. The general case

If a system is neither cyclic (including staurocyclic) nor stauric, it has the following properties, by elimination:
- N has some square factor ;

- μ and φ have a common factor ;
- $\mu . \varphi > N$.

The smallest examples with Abelian group are:
a) $G = C_4 \times C_2$, *m* = (1 2 3 4) (5 6 7 8), *f* = (1 5 3 7) (2 6 4 8)
b) $G = C_6 \times C_2$, *m* = (1 2 3 4 5 6) (7 8 9 10 11 12), *f* = (1 7 3 9 5 11) (2 8 4 10 6 12)
c) $G = C_6 \times C_2$, *m* = (1 2 3 4 5 6) (7 8 9 10 11 12), *f* = (1 7 5 11 3 9) (2 8 6 12 4 10)

All three are self-dual, that is, their properties (especially their Cayley diagram) are left unchanged by an exchange of *m* and *f*. This is uncommon among other Abelian systems.
- Holocyclic systems are self-dual iff $x = N-1$ (i.e. $m . f = I$).
- Staurocyclic systems can't be self-dual, since $\mu = \varphi$ is impossible (remember that they must be relatively prime).
- Stauric systems are self-dual iff $\mu = \varphi$.

7. Open problems

Systems a) and b) from section 6.2.2. satisfy both equations (2) and (3), and might well be the smallest representatives of an infinite class. More generally, new infinite subclass(es) could well arise when N is the square of a nonprime (16 being the smallest such N, but a very intricate case) or when $G = C_a \times C_b$, with neither a nor b prime (24 being the smallest possible order).

All systems we found, in which both *f* and *m* are of order N/2, are self-dual. This might be a general property, but we didn't prove it.

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